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LETTER TO THE EDITOR

Heisenberg–Weyl algebras of symmetric and antisymmetric bosons

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Abstract. All the polynomials in symmetric ({2}) and antisymmetric ({11}) u(n) bosons are constructed in u(n) bases and u(n)-reduced matrix elements for the bosons between polynomial basis states are computed. Applications to representation theory of Lie groups, paired-fermion and boson physics are briefly discussed.

The nth Heisenberg-Weyl algebra $hw(n)$ is characterised by the commutation relations

$$[\nabla_i, z_j] = \delta_{ij} \qquad i, j = 1, \ldots, n$$

where, in the Bargmann representation, z_i is realised as a complex variable and $\nabla_i = \partial/\partial z_i$. This algebra is widely used in physics and mathematics. In field theory, for example, the raising (z_i) and lowering (∇_i) operators are interpreted as creation and annihilation operators of n-component vector bosons. They are also interpreted as the quanta (phonons) of many-body and lattice vibrations. Thus, one often needs to classify many-boson states and determine the matrix elements of the boson operators between such states.

This is simple for the vector boson because the boson raising operators span the fundamental {1} irrep of a u(n) algebra. Consequently, the N-boson states span the fully symmetric irrep $\{N\}$ of u(n) and one has the well known u(n)-reduced matrix elements

$$\langle\{N+1\}\|z\|\{N\}\rangle = (N+1)^{1/2}.$$

In this letter, we consider tensor bosons, which carry irreps {2} and {11} of u(n), respectively. The components of the corresponding tensor bosons are now conveniently labelled by double indices (z_{ij}; $1 \leq i, j \leq n$) where $z_{ij} = z_{ji}$ for {2} and $z_{ij} = -z_{ji}$ for {11}. For brevity, we therefore refer to them as symmetric and antisymmetric bosons, respectively. They are the raising operators of $hw(n(n+1)/2)$ and $hw(n(n-1)/2)$ Heisenberg-Weyl algebras, respectively, and the N-boson states carry fully symmetric irreps $\{N\}$ of u($n(n+1)/2$) and u($n(n-1)/2$), respectively. However, we wish to classify them by means of the generally much smaller u(n) Lie algebra.

Tensor bosons arise in physics as composites of more elementary objects, e.g. quark or Cooper (fermion) pairs. An understanding of their algebraic properties is therefore needed in the application of boson models in physics. They also appear in the contraction of classical Lie algebras. Thus, whereas the u($n+1$) algebra contains raising operators (in addition to those of its u(n) subalgebra) which contract to the

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components of a vector boson, the sp(n) and so($2n$) algebras contain raising operators which contract to {2}- and {11}-tensor bosons, respectively. The determination of explicit formulae for boson matrix elements is therefore of major importance in representation theory. In addition, the corresponding D and B series feature in the study of branching rules and Kronecker products both for compact (King 1975, Black *et al* 1983) and non-compact (Rowe *et al* 1985b) groups.

Recent developments in vector coherent state theory (Rowe 1984, Deenen and Quesne 1984, Rowe *et al* 1985a) have emphasised the importance of these contractions in the representation theory of the classical Lie algebras. In particular, it has been demonstrated that, starting from irreps of u(n), one can induce irreps of classical Lie algebras and infer their explicit matrix representations in terms of u(n) Wigner and Racah coefficients and the matrix elements of {1}-, {2}- and {11}-tensor bosons. This construction has been given explicitly by Hecht *et al* (1987) for su($n+1$) $\supset$ u(n), by Rowe (1984) and Rowe *et al* (1985a) for sp($2n, \mathfrak{R}$) $\supset$ u(n) and by Hecht and Elliott (1985) for sp(4) $\supset$ u(2). It has been given by Hecht (1985) for so(8) $\supset$ u(4), Le Blanc and Rowe (1986) for so*($2n$) $\supset$ u(n) and so(n, 2) $\supset$ so(n) and by Rowe and Carvalho (1986) for so($2n$) $\supset$ u(n).

All of these results need matrix elements of the above-mentioned rank one and two tensor bosons, some of which are already known. (The simple $n=2$ symmetric case has been reviewed by Le Blanc and Rowe (1986). The $n=3$ symmetric case has been studied by Quesne (1981) and Rosensteel and Rowe (1983) and is applicable to the symplectic model of nuclear collective motions of Rosensteel and Rowe (1980) and to the interacting boson approximation of Arima and Iachello (1983). The $n=3$ antisymmetric case has been given by Hecht (1987).) The other necessary input for their use, namely a knowledge of the u(n) Wigner–Racah calculus, has been developed by Biedenharn and Louck (see Louck 1970) and further developed and identified with the u(n) representation theory in the context of vector coherent state theory by Le Blanc and Hecht (1987).

First we consider the symmetric case. The symmetric Bargmann variables

$$z_{ij} = z_{ji} \qquad 1 \leq i, j \leq n \tag{1}$$

obey the commutation relations

$$[\nabla_{ij}, z_{kl}] = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{2}$$

where $\nabla_{ij} = (z_{ij})^\dagger$ with respect to the Bargmann measure.

The set of operators

$$E_{ij} = z_{ia}\nabla_{aj} \tag{3}$$

with summation over repeated indices, generates a u(n) Lie algebra obeying the usual commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}. \tag{4}$$

One verifies that the set (z_{ij}) spans the u(n) symmetrical irrep {2}.

It is known (Thrall 1942) that the set of all polynomials in the (z_{ij}) reduces under u(n) to a direct sum of all tensor irreps labelled by partitions $\{d\} = \{d_1, d_2\ , \ldots, d_n\}$ in n parts belonging to D, the set of partitions of even integer parts.

We now show that the polynomial corresponding to the highest weight component of the irreducible tensor representation $\{d\}$ is given by

$$\langle z|\{d\}\text{hw}\rangle = N(\{d\}) z_{11}^{(d_1-d_2)/2} \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}^{(d_2-d_3)/2} \cdots \begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{vmatrix}^{d_n/2} \tag{5a}$$

where $N(\{d\})$ is the normalisation factor given by

$$N(\{d\}) = \left[\prod_{i<j}^{n} \left(\frac{(d_i - d_j + j - i)!!}{(d_i - d_j + j - i - 1)!!}\right) \prod_{i=1}^{n} \left(\frac{1}{(d_i + n - i)!!}\right)\right]^{1/2}. \tag{5b}$$

That the expression $(5a)$ is highest weight is easily verified by the vanishing action of the raising operators E_{ij}, $i<j$, on it. The factor $N(\{d\})$ is easily calculated using a generalisation of the Cappelli operator identity to the case of the symmetric bosons (Weyl 1946) which states that, for $n = k$,

$$\begin{vmatrix} \nabla_{11} & \nabla_{12} & \cdots & \nabla_{1k} \\ \nabla_{21} & \nabla_{22} & \cdots & \nabla_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_{k1} & \nabla_{k2} & \cdots & \nabla_{kk} \end{vmatrix} \begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1k} \\ z_{21} & z_{22} & \cdots & z_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k1} & z_{k2} & \cdots & z_{kk} \end{vmatrix} = \begin{vmatrix} E_{11}+k+1 & E_{12} & \cdots & E_{1k} \\ E_{21} & E_{22}+k & \cdots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{k1} & E_{k2} & \cdots & E_{kk}+2 \end{vmatrix} \tag{6}$$

and which, for $n = k = 1$, reduces to

$$\nabla_{11} z_{11} = z_{11} \nabla_{11} + 2.$$

It is understood that in the expansion of the determinant on the right-hand side of (6) the products of generators are ordered by increasing column index.

Defining the unnormalised (round) ket

$$|\{d\}\text{hw}) = z_{11}^{(d_1-d_2)/2} \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}^{(d_2-d_3)/2} \cdots \begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{vmatrix}^{d_n/2}$$

we have that

$$\begin{aligned} &(\{d_1, d_2, \ldots, d_n\}\text{hw}|\{d_1, d_2, \ldots, d_n\}\text{hw}) \\ &\quad = (\{d_1-2, d_2-2, \ldots, d_n-2\}\text{hw}| \\ &\quad \times \begin{vmatrix} \nabla_{11} & \nabla_{12} & \cdots & \nabla_{1m} \\ \nabla_{21} & \nabla_{22} & \cdots & \nabla_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_{ml} & \nabla_{m2} & \cdots & \nabla_{mm} \end{vmatrix} \begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\ z_{21} & z_{22} & \cdots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mm} \end{vmatrix} \\ &\quad \times |\{d_1-2, d_2-2, \ldots, d_n-2\}\text{hw}) \end{aligned} \tag{7a}$$

which, using the Capelli identity and since the ket is of highest weight, reduces to

$$
\begin{aligned}
&(\{d_1, d_2, \ldots, d_n\}\mathrm{hw}|\{d_1, d_2, \ldots, d_n\}\mathrm{hw}) \\
&\qquad = (\{d_1-2, d_2-2, \ldots, d_n-2\}\mathrm{hw}|[(E_{11}+n+1)(E_{22}+n)\ldots \\
&\qquad\quad \times (E_{nn}+2)]|\{d_1-2, d_2-2, \ldots, d_n-2\}\mathrm{hw}) \\
&\qquad = [(d_1+n-1)(d_2+n-2)\ldots(d_n)] \\
&\qquad\quad \times (\{d_1-2, d_2-2, \ldots, d_n-2\}\mathrm{hw}|\{d_1-2, d_2-2, \ldots, d_n-2\}\mathrm{hw}). \qquad (7b)
\end{aligned}
$$

An iteration of the above steps until d_n, $d_{n-1}-d_n$, etc, vanish verifies (5*b*).

We now compute u(n)-reduced matrix elements of the elementary {2} tensor operator (1) between states belonging to the tensor representations $\{d\}$ and $\{d+2\Delta^{(1)}(k)\}$, where $\Delta^{(1)}(k)$ is the n vector $(00\ldots010\ldots0)$ with null entries everywhere except for the numeral 1 in its kth entry.

Using the Cappelli identity, one determines the matrix element

$$
\begin{aligned}
&\langle\{d+2\Delta^{(1)}(k)\}\mathrm{hw}|(z_{kk}/\sqrt{2})|\{d\}\mathrm{hw}\rangle \\
&\qquad = \langle\{d\}\mathrm{hw}|(\nabla_{kk}/\sqrt{2})|\{d+2\Delta^{(1)}(k)\}\mathrm{hw}\rangle \\
&\qquad = \frac{(d_k+n+2-k)}{\sqrt{2}}\,\frac{N(\{d+2\Delta^{(1)}(k)\})}{N(\{d\})}\left[\prod_{j=k+1}^{n}\left(\frac{d_k-d_j+j-k+1}{d_k-d_j+j-k+2}\right)\right] \qquad (8)
\end{aligned}
$$

with no sum on k. The Wigner coefficient needed to isolate the reduced matrix element from the matrix element (8) is obtained following the pattern calculus of Biedenharn and Louck (1968) and is given by

$$
\left[\prod_{j=1}^{k-1}\left(\frac{d_k-d_j+j-k+2}{d_k-d_j+j-k}\right)\right]^{1/2}. \qquad (9)
$$

Dividing the right-hand side of (8) by (9), we find

$$
\langle\{d+2\Delta^{(1)}(k)\}\|z\|\{d\}\rangle = \left[\tfrac{1}{2}(d_k+n+2-k)\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\frac{d_k-d_j+j-k+1}{d_k-d_j+j-k+2}\right)\right]^{1/2}. \qquad (10)
$$

Since the set of all the irreducible tensorial representations $\{d\}$ in the symmetric bosons with $d_1+d_2+\ldots+d_n=2d_s$ fixed spans the one-rowed unirrep $\{d_s\}$ of the u$(n(n+1)/2)$ Lie algebra spanned by the generators $E_{ab,cd}=z_{ab}\nabla_{cd}$, the reduced matrix elements (10) must satisfy the sum rule

$$
\begin{aligned}
&\sum_{k=1}^{n}\langle\{d\}\|z\|\{d-2\Delta^{(1)}(k)\}\rangle^2 \\
&\qquad = \sum_{k=1}^{n}\left[\tfrac{1}{2}(d_k+n-k)\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\frac{d_k-d_j+j-k-1}{d_k-d_j+j-k}\right)\right] \qquad (11a) \\
&\qquad = \tfrac{1}{2}\sum_{j=1}^{n} d_j. \qquad (11b)
\end{aligned}
$$

The sum in (11*a*) is a sum over the residues of the complex function

$$
f(Z) = -\tfrac{1}{2}Z\prod_{j=1}^{n}\left(\frac{Z-p_{jn}-1}{Z-p_{jn}}\right) \qquad p_{ij}=d_i+j-i \qquad (12)
$$

a sum which is equal to minus the residue of this same function at infinity. The latter residue is defined by the residue of the function

$$g(Z) = -\frac{1}{Z^2} f\left(\frac{1}{Z}\right) \tag{13}$$

at zero which is found to be given by minus the sum in equation (11*b*), thus verifying equation (10).

Next we consider the antisymmetric case. The antisymmetric Bargmann variables

$$z_{ij} = -z_{ji} \qquad 1 \leqslant i, j \leqslant n \tag{14}$$

obey the commutation relations

$$[\nabla_{ij}, z_{kl}] = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{15}$$

where $\nabla_{ij} = -\nabla_{ji} = (z_{ij})^{\dagger}$ with respect to the Bargmann measure.

The set of operators

$$E_{ij} = -z_{ia}\nabla_{aj} \tag{16}$$

also generates a u(n) Lie algebra obeying the commutations relations (4). The set (z_{ij}) now spans an antisymmetrical u(n) irrep {11}.

The set of all polynomials in (z_{ij}) now reduces under u(n) to a direct sum of all tensor irreps labelled by partitions $\{b\} \in B$, the set of partitions conjugate to the set D of partitions of even integer parts (King 1975, Black *et al* 1983). A generic partition $\{b\} \in B$ will thus be labelled by $\{b_1, b_2, \ldots, b_n\}$ with

$$b_1 = b_2 \geqslant b_3 = b_4 \geqslant \ldots \geqslant b_{2m-1} = b_{2m} \text{ for } n = 2m,\ b_{2m+1} = 0 \text{ for } n = 2m+1. \tag{17}$$

We now show that the polynomial corresponding to the highest weight component of the irreducible tensor representation $\{b\}$ is given by

$$\langle z|\{b\}\text{hw}\rangle = N(\{b\})\left(\frac{\varepsilon_{\pi_1\pi_2} z_{\pi_1\pi_2}}{1!2^1}\right)^{b_1-b_3}_{\pi\leqslant 2}\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4} z_{\pi_1\pi_2} z_{\pi_3\pi_4}}{2!2^2}\right)^{b_3-b_5}_{\pi\leqslant 4}\cdots$$
$$\times\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4\ldots\pi_{2m-1}\pi_{2m}} z_{\pi_1\pi_2} z_{\pi_3\pi_4}\cdots z_{\pi_{2m-1}\pi_{2m}}}{m!2^m}\right)^{b_{2m-1}}_{\pi\leqslant 2m} \tag{18a}$$

where the normalisation factor $N(\{b\})$ is given by

$$N(\{b\}) = \left(\frac{\Pi^m_{i<j}[(b_{2i-1} - b_{2j-1} + 2j - 2i)(b_{2i-1} - b_{2j-1} + 2j - 2i - 1)]}{\Pi^m_{i=1}(b_{2i-1} + 2m - 2i)!}\right)^{1/2} \tag{18b}$$

and where $\varepsilon_{\pi_1\pi_2\ldots\pi_n}$ is the totally antisymmetric tensor in n dimensions with $\varepsilon_{12\ldots n} = +1$. That the expression (18*a*) is of highest weight is easily seen from the (antisymmetric) tensorial properties of the elementary polynomials

$$\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4\ldots\pi_{2a-1}\pi_{2a}} z_{\pi_1\pi_2} z_{\pi_3\pi_4}\cdots z_{\pi_{2a-1}\pi_{2a}}}{a!2^a}\right)_{\pi\leqslant 2a}$$

from which it is built.

Although no equivalent to the Cappelli identity is known for the antisymmetric boson case, it can be inferred that the weaker identity

$$\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4\ldots\pi_{2m-1}\pi_{2m}}\nabla_{\pi_1\pi_2}\nabla_{\pi_3\pi_4}\ldots\nabla_{\pi_{2m-1}\pi_{2m}}}{m!2^m}\right)_{\pi\leqslant 2m}$$
$$\times\left[\left(\frac{\varepsilon_{\pi_1\pi_2}z_{\pi_1\pi_2}}{1!2^1}\right)_{\pi\leqslant 2}^{b_1-b_3}\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4}z_{\pi_1\pi_2}z_{\pi_3\pi_4}}{2!2^2}\right)_{\pi\leqslant 4}^{b_3-b_5}\ldots\right.$$
$$\left.\times\left(\frac{\varepsilon_{\pi_1\pi_2\ldots\pi_{2m}}z_{\pi_1\pi_2}z_{\pi_3\pi_4}\ldots z_{\pi_{2m-1}\pi_{2m}}}{m!2^m}\right)_{\pi\leqslant 2m}^{b_{2m-1}}\right]$$
$$=[(E_{11}+2m-1)(E_{33}+2m-3)\ldots(E_{2m-1,2m-1}+1)]$$
$$\times\left[\left(\frac{\varepsilon_{\pi_1\pi_2}z_{\pi_1\pi_2}}{1!2^1}\right)_{\pi\leqslant 2}^{b_1-b_3}\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4}z_{\pi_1\pi_2}z_{\pi_3\pi_4}}{2!2^2}\right)_{\pi\leqslant 4}^{b_3-b_5}\ldots\right.$$
$$\left.\times\left(\frac{\varepsilon_{\pi_1\pi_2\ldots\pi_{2m}}z_{\pi_1\pi_2}z_{\pi_3\pi_4}\ldots z_{\pi_{2m-1}\pi_{2m}}}{m!2^m}\right)_{\pi\leqslant 2m}^{b_{2m-1}-1}\right] \tag{19}$$

holds, from which one easily derives the normalisation coefficient $N(\{b\})$.

We now compute the u(n)-reduced matrix elements of the elementary $\{11\}$ tensor operator (14) between states belonging to the tensor representations $\{b\}$ and $\{b+\Delta^{(11)}(k)\}$ where $\Delta^{(11)}(k)$ is the n vector $(00\ldots0110\ldots0)$ with null entries everywhere except for the numeral 1 in its $(2k-1)$th and $(2k)$th entries.

Using (19) and the identity

$$\nabla_{2a-1,2a}\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4\ldots\pi_{2a-1}\pi_{2a}}z_{\pi_1\pi_2}z_{\pi_3\pi_4}\ldots z_{\pi_{2a-1}\pi_{2a}}}{a!2^a}\right)_{\pi\leqslant 2a}$$
$$=\left(\frac{\varepsilon_{\pi_1\pi_2\pi_3\pi_4\ldots\pi_{2a-3}\pi_{2a-2}}z_{\pi_1\pi_2}z_{\pi_3\pi_4}\ldots z_{\pi_{2a-3}\pi_{2a-2}}}{(a-1)!2^{a-1}}\right)_{\pi\leqslant 2a-2} \tag{20}$$

we easily determine the matrix element

$$\langle\{b+\Delta^{(11)}(k)\}\mathrm{hw}|z_{2k-1,2k}|\{b\}\mathrm{hw}\rangle$$
$$=\langle\{b\}\mathrm{hw}|\nabla_{2k-1,2k}|\{b+\Delta^{(11)}(k)\}\mathrm{hw}\rangle$$
$$=(b_{2k-1}-b_{2k+1}+1)\frac{N(\{b+\Delta^{(11)}(k)\})}{N(\{b\})}$$
$$\times\left[\prod_{j=k+1}^{m}\left(\frac{b_{2k-1}-b_{2j+1}+2j-2k+1}{b_{2k-1}-b_{2j-1}+2j-2k+1}\right)\right]. \tag{21}$$

The Wigner coefficient needed to isolate the reduced matrix element from (21) is obtained once more through the use of the pattern calculus and is given by

$$\left[\prod_{j=1}^{k-1}\left(\frac{b_{2k-1}-b_{2j-1}+2j-2k+1}{b_{2k-1}-b_{2j-1}+2j-2k-1}\right)\prod_{j=1}^{k-1}\left(\frac{b_{2k-1}-b_{2j-1}+2j-2k+2}{b_{2k-1}-b_{2j-1}+2j-2k}\right)\right]^{1/2}. \tag{22}$$

Dividing the right-hand side of (21) by (22), we find

$$\langle\{b+\Delta^{(11)}(k)\}\|z\|\{b\}\rangle$$
$$=\left[(b_{2k-1}+2m-2k+1)\prod_{\substack{j=1\\j\neq k}}^{m}\left(\frac{b_{2k-1}-b_{2j-1}+2j-2k-1}{b_{2k-1}-b_{2j-1}+2j-2k+1}\right)\right]^{1/2}. \tag{23}$$

Since the set of all the irreducible tensorial representations $\{b\}$ in the antisymmetric bosons with $b_1+b_3+\ldots+b_{2m-1}=b_a$ fixed spans the one-rowed unirrep $\{b_a\}$ of the $\mathrm{u}(n(n-1)/2)$ Lie algebra generated by $E_{ab,cd}=z_{ab}\nabla_{cd}$, the reduced matrix elements (23) must satisfy the sum rule

$$\sum_{k=1}^{m}\langle\{b\}\|z\|\{b-\Delta^{(11)}(k)\}\rangle^2 = \sum_{k=1}^{m}\left[(b_{2k-1}+2m-2k)\prod_{\substack{j=1\\ j\neq k}}^{m}\left(\frac{b_{2k-1}-b_{2j-1}+2j-2k-2}{b_{2k-1}-b_{2j-1}+2j-2k}\right)\right] \qquad (24a)$$

$$=\sum_{j=1}^{m} b_{2j-1}. \qquad (24b)$$

The sum in (24a) is a sum over the residues of the complex function

$$f(Z)=-\tfrac{1}{2}Z\prod_{j=1}^{m}\left(\frac{Z-p_{2j-1,2m-1}-2}{Z-p_{2j-1,2m-1}}\right) \qquad p_{2j-1,2m-1}=b_{2j-1}+(2m-1)-(2j-1)$$

a sum which is equal to minus the residue of this same function at infinity. One easily finds the latter residue to be given by minus the sum in equation (24b) thus verifying equation (23).

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